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AUTHOR(S):

Ito, Kenichi; Jensen, Arne

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Resolvent expansion for the Schrödinger operator on a graph with infinite rays

Kenichi ITO* & Arne JENSEN†

In this article we report on the authors' recent work [IJ3] on an expansion of the resolvent for the Schrödinger operator on a graph with rays. We obtain precise expressions for the first few coefficients of the expansion around the threshold 0 in terms only of the generalized eigenfunctions. This in particular justifies the natural definition of threshold resonances for the generalized eigenfunctions solely by the growth rate at infinity.

1 The free operator

In this section we define a *graph with rays*, and fix our free operator H_0 on it. Here we denote the set of vertices by G , and the set of edges by E_G , hence we consider the graph (G, E_G) . We sometimes call it simply the graph G . The free operator H_0 is defined as a direct sum of the free Dirichlet Schrödinger operators on a finite part and rays, being different from the graph Laplacian $-\Delta_G$.

Let (K, E_0) be a connected, finite, undirected and simple graph, without loops or multiple edges, and let (L_α, E_α) , $\alpha = 1, \dots, N$, be N copies of the discrete half-line, i.e.

$$L_\alpha = \mathbb{N} = \{1, 2, \dots\}, \quad E_\alpha = \{\{n, n+1\}; n \in L_\alpha\}.$$

We construct the graph (G, E_G) by jointing (L_α, E_α) to (K, E_0) at a vertex $x_\alpha \in K$ for $\alpha = 1, \dots, N$:

$$G = K \cup L_1 \cup \dots \cup L_N, \\ E_G = E_0 \cup E_1 \cup \dots \cup E_N \cup \{\{x_1, 1^{(1)}\}, \dots, \{x_N, 1^{(N)}\}\}.$$

*Department of Mathematics, Graduate School of Science, Kobe University, 1-1, Rokkodai, Nada-ku, Kobe 657-8501, Japan. E-mail: ito-ken@math.kobe-u.ac.jp.

†Department of Mathematical Sciences, Aalborg University, Skjernvej 4A, DK-9220 Aalborg Ø, Denmark. E-mail: matarne@math.aau.dk.

Here we distinguished 1 of L_α by a superscript: $1^{(\alpha)} \in L_\alpha$. Note that two different half-lines (L_α, E_α) and (L_β, E_β) , $\alpha \neq \beta$, could be jointed to the same vertex $x_\alpha = x_\beta \in K$.

Let h_0 be the free *Dirichlet* Schrödinger operators on K : For any function $u: K \rightarrow \mathbb{C}$ we define

$$(h_0 u)[x] = \sum_{\{x,y\} \in E_0} (u[x] - u[y]) + \sum_{\alpha=1}^N s_\alpha[x] u[x] \quad \text{for } x \in K,$$

where $s_\alpha[x] = 1$ if $x = x_\alpha$ and $s_\alpha[x] = 0$ otherwise. Note that the Dirichlet boundary condition is considered being set on the *boundaries* $1^{(\alpha)} \in L_\alpha$ outside K . Similarly, for $\alpha = 1, \dots, N$ let h_α be the free Dirichlet Schrödinger operators on L_α : For any function $u: L_\alpha \rightarrow \mathbb{C}$ we define

$$(h_\alpha u)[n] = \begin{cases} 2u[1] - u[2] & \text{for } n = 1, \\ 2u[n] - u[n+1] - u[n-1] & \text{for } n \geq 2. \end{cases}$$

Then we define the free operator H_0 on G as a direct sum

$$H_0 = h_0 \oplus h_1 \oplus \dots \oplus h_N, \quad (1.1)$$

according to a direct sum decomposition

$$F(G) = F(K) \oplus F(L_1) \oplus \dots \oplus F(L_N),$$

where $F(X) = \{u: X \rightarrow \mathbb{C}\}$ denotes the set of all the functions on a space X .

In the definition (1.1) interactions between K and L_α are absent, and the free operator H_0 does not coincide with the graph Laplacian $-\Delta_G$ defined as

$$(-\Delta_G u)[x] = \sum_{\{x,y\} \in E_G} (u[x] - u[y]).$$

In fact, we can write

$$-\Delta_G = H_0 + J, \quad J = - \sum_{\alpha=1}^N \left(|s_\alpha\rangle \langle f_\alpha| + |f_\alpha\rangle \langle s_\alpha| \right), \quad (1.2)$$

where $f_\alpha[x] = 1$ if $x = 1^{(\alpha)}$ and $f_\alpha[x] = 0$ otherwise. The operator H_0 is actually simpler and more useful than $-\Delta_G$, since it does not have a zero eigenvalue or a zero resonance, and the asymptotic expansion of its resolvent around 0 does not have a singular part. This fact effectively simplifies the expansion procedure for the perturbed resolvent, and enables us to obtain more precise expressions for the coefficients than those in [IJ1]. The interaction J is a special case of general perturbations considered in Assumption 2.1, see Proposition 2.2. Hence the graph Laplacian $-\Delta_G$ can be treated as a perturbation of the free operator H_0 .

2 The perturbed operator

In this section we introduce our class of perturbations. We also provide a simple classification of threshold types in terms of the growth rate of the generalized eigenfunctions. This classification will be justified by our main results presented in Section 3.

Set for $s \in \mathbb{R}$

$$\begin{aligned}\mathcal{L}^s &= \ell^{1,s}(G) = (\ell^1(K)) \oplus (\ell^{1,s}(L_1)) \oplus \cdots \oplus (\ell^{1,s}(L_N)), \\ (\mathcal{L}^s)^* &= \ell^{\infty,-s}(G) = (\ell^\infty(K)) \oplus (\ell^{1,s}(L_1)) \oplus \cdots \oplus (\ell^{1,s}(L_N)),\end{aligned}$$

where for $\alpha = 1, \dots, N$

$$\begin{aligned}\ell^{1,s}(L_\alpha) &= \left\{ x: L_\alpha \rightarrow \mathbb{C}; \sum_{n \in L_\alpha} (1 + n^2)^{s/2} |x[n]| < \infty \right\}, \\ \ell^{\infty,-s}(L_\alpha) &= \left\{ x: L_\alpha \rightarrow \mathbb{C}; \sup_{n \in L_\alpha} (1 + n^2)^{-s/2} |x[n]| < \infty \right\}.\end{aligned}$$

We consider the following class of perturbations, cf. [JN1, IJ1, IJ2].

Assumption 2.1. Assume that $V \in \mathcal{B}(\mathcal{H})$ is self-adjoint, and that there exist an injective operator $v \in \mathcal{B}(\mathcal{K}, \mathcal{L}^\beta)$ with $\beta \geq 1$ and a self-adjoint unitary operator $U \in \mathcal{B}(\mathcal{K})$, both defined on some abstract Hilbert space \mathcal{K} , such that

$$V = vUv^* \in \mathcal{B}((\mathcal{L}^\beta)^*, \mathcal{L}^\beta).$$

We note that V is compact on \mathcal{H} under Assumption 2.1. Let us provide a criterion for Assumption 2.1 in terms of weighted ℓ^2 -spaces. We use the standard weighted space notation such as $\ell^{2,s}(G)$, $s \in \mathbb{R}$.

Proposition 2.2. Assume that $V \in \mathcal{B}(\mathcal{H})$ is self-adjoint, and that it extends to an operator in $\mathcal{B}(\ell^{2,-\beta-1/2-\epsilon}(G), \ell^{2,\beta+1/2+\epsilon}(G))$ for some $\beta \geq 1$ and $\epsilon > 0$. Then V satisfies Assumption 2.1 for the same β .

By this criterion we can see that the interaction J from (1.2) satisfies Assumption 2.1. For another criterion for Assumption 2.1 we refer to [IJ1, Appendix B].

Under Assumption 2.1 we let

$$H = H_0 + V,$$

and consider the solutions to the zero eigen-equation $H\Psi = 0$ in the largest space where it can be defined. Define the *generalized zero eigenspace* $\tilde{\mathcal{E}}$ as

$$\tilde{\mathcal{E}} = \{\Psi \in (\mathcal{L}^\beta)^*; H\Psi = 0\}.$$

Let $\mathbf{n}^{(\alpha)} \in (\mathcal{L}^1)^*$, $\mathbf{1}^{(\alpha)} \in (\mathcal{L}^0)^*$ be the functions defined as

$$\mathbf{n}^{(\alpha)}[x] = \begin{cases} m & \text{for } x = m \in L_\alpha, \\ 0 & \text{for } x \in G \setminus L_\alpha, \end{cases} \quad \mathbf{1}^{(\alpha)}[x] = \begin{cases} 1 & \text{for } x \in L_\alpha, \\ 0 & \text{for } x \in G \setminus L_\alpha, \end{cases}$$

respectively, and abbreviate the spaces spanned by these functions as

$$\mathbb{C}\mathbf{n} = \mathbb{C}\mathbf{n}^{(1)} \oplus \cdots \oplus \mathbb{C}\mathbf{n}^{(N)}, \quad \mathbb{C}\mathbf{1} = \mathbb{C}\mathbf{1}^{(1)} \oplus \cdots \oplus \mathbb{C}\mathbf{1}^{(N)}.$$

We can show that under Assumption 2.1 with $\beta \geq 1$ the generalized eigenfunctions have specific asymptotics:

$$\tilde{\mathcal{E}} \subset \mathbb{C}\mathbf{n} \oplus \mathbb{C}\mathbf{1} \oplus \mathcal{L}^{\beta-2}.$$

With this asymptotics we consider the following subspaces:

$$\mathcal{E} = \tilde{\mathcal{E}} \cap (\mathbb{C}\mathbf{1} \oplus \mathcal{L}^{\beta-2}), \quad \mathbf{E} = \tilde{\mathcal{E}} \cap \mathcal{L}^{\beta-2}.$$

A function in $\tilde{\mathcal{E}} \setminus \mathcal{E}$ should be called a *non-resonance eigenfunction*, one in $\mathcal{E} \setminus \mathbf{E}$ a *resonance eigenfunction*, and one in \mathbf{E} a *bound eigenfunction*, but we shall often call them *generalized eigenfunctions* or simply *eigenfunctions*.

Let us introduce the same classification of threshold as in [IJ1, Definition 1.6].

Definition 2.3. The threshold $z = 0$ is said to be

1. a *regular point*, if $\mathcal{E} = \mathbf{E} = \{0\}$;
2. an *exceptional point of the first kind*, if $\mathcal{E} \supsetneq \mathbf{E} = \{0\}$;
3. an *exceptional point of the second kind*, if $\mathcal{E} = \mathbf{E} \supsetneq \{0\}$;
4. an *exceptional point of the third kind*, if $\mathcal{E} \supsetneq \mathbf{E} \supsetneq \{0\}$.

It should be noted here that there is a dimensional relation:

$$\dim(\tilde{\mathcal{E}}/\mathcal{E}) + \dim(\mathcal{E}/\mathbf{E}) = N, \quad 0 \leq \dim \mathbf{E} < \infty,$$

the former of which reflects a certain topological stability of the non-decaying eigenspace under small perturbations.

We can also show that for any $\Psi_1 \in \tilde{\mathcal{E}}$ and $\Psi_2 \in \mathcal{E}$, if we let

$$\Psi_1 - \sum_{\alpha=1}^N c_\alpha^{(1)} \mathbf{n}^{(\alpha)} \in \mathbb{C}\mathbf{1} \oplus \mathcal{L}^{\beta-2}, \quad \Psi_2 - \sum_{\alpha=1}^N c_\alpha^{(2)} \mathbf{1}^{(\alpha)} \in \mathcal{L}^{\beta-2},$$

then these coefficients are orthogonal:

$$\sum_{\alpha=1}^N \bar{c}_{\alpha}^{(2)} c_{\alpha}^{(1)} = 0.$$

By this fact it would be natural to introduce *orthogonality* in $\tilde{\mathcal{E}}$ in terms of the asymptotics, and accordingly define the *generalized orthogonal projections*. We use $\langle \cdot, \cdot \rangle$ to denote the duality between \mathcal{L}^s and $(\mathcal{L}^s)^*$. If $\beta \geq 2$ then $\langle \Phi, \Psi \rangle$ is defined for $\Phi \in \mathbf{E}$ and $\Psi \in \mathcal{E}$. If we only assume $\beta \geq 1$ then we must assume $\Phi \cdot \Psi \in \mathcal{L}^0$ to justify the notation $\langle \Phi, \Psi \rangle$. Here $(\Phi \cdot \Psi)[n] = \Psi[n]\Phi[n]$, $n \in G$, is the pointwise product.

Definition 2.4. We call a subset $\{\Psi_{\gamma}\}_{\gamma} \subset \mathcal{E}$ a *resonance basis*, if the set $\{[\Psi_{\gamma}]\}_{\gamma}$ of representatives forms a basis in \mathcal{E}/\mathbf{E} . It is said to be *orthonormal*, if

1. for any γ and $\Psi \in \mathbf{E}$ one has $\bar{\Psi} \cdot \Psi_{\gamma} \in \mathcal{L}^0$ and $\langle \Psi, \Psi_{\gamma} \rangle = 0$;
2. there exists an orthonormal system $\{c^{(\gamma)}\}_{\gamma} \subset \mathbb{C}^N$ such that for any γ

$$\Psi_{\gamma} - \sum_{\alpha=1}^N c_{\alpha}^{(\gamma)} \mathbf{1}^{(\alpha)} \in \mathcal{L}^{\beta-2}.$$

The *orthogonal resonance projection* \mathcal{P} is defined as

$$\mathcal{P} = \sum_{\gamma} |\Psi_{\gamma}\rangle \langle \Psi_{\gamma}|.$$

Definition 2.5. We call a basis $\{\Psi_{\gamma}\}_{\gamma} \subset \mathbf{E}$ a *bound basis* to distinguish it from a resonance basis. It is said to be *orthonormal*, if for any γ and γ' one has $\bar{\Psi}_{\gamma'} \cdot \Psi_{\gamma} \in \mathcal{L}^0$ and

$$\langle \Psi_{\gamma'}, \Psi_{\gamma} \rangle = \delta_{\gamma\gamma'}.$$

The *orthogonal bound projection* \mathbf{P} is defined as

$$\mathbf{P} = \sum_{\gamma} |\Psi_{\gamma}\rangle \langle \Psi_{\gamma}|.$$

We remark that the above orthogonal projections \mathcal{P} and \mathbf{P} are independent of choice of orthonormal bases.

3 Main results

In this section we present the main theorems of [IJ3] classifying the resolvent expansions according to threshold types given in Definition 2.3. In the statements below we have to impose different assumptions on the parameter β depending on threshold types. For simplicity we state the results only for integer values of β , but an extension to general β is straightforward.

We set

$$R(\kappa) = (H + \kappa^2)^{-1} \text{ for } -\kappa^2 \notin \sigma(H), \quad \mathcal{B}^s = \mathcal{B}(\mathcal{L}^s, (\mathcal{L}^s)^*).$$

Theorem 3.1. *Assume that the threshold 0 is a regular point, and that Assumption 2.1 is fulfilled for some integer $\beta \geq 2$. Then*

$$R(\kappa) = \sum_{j=0}^{\beta-2} \kappa^j G_j + \mathcal{O}(\kappa^{\beta-1}) \quad \text{in } \mathcal{B}^{\beta-2}$$

with $G_j \in \mathcal{B}^{j+1}$ for j even, and $G_j \in \mathcal{B}^j$ for j odd. The coefficients G_j can be computed explicitly. In particular,

$$G_{-2} = \mathcal{P} = 0, \quad G_{-1} = \mathcal{P} = 0.$$

Theorem 3.2. *Assume that the threshold 0 is an exceptional point of the first kind, and that Assumption 2.1 is fulfilled for some integer $\beta \geq 3$. Then*

$$R(\kappa) = \sum_{j=-1}^{\beta-4} \kappa^j G_j + \mathcal{O}(\kappa^{\beta-3}) \quad \text{in } \mathcal{B}^{\beta-1}$$

with $G_j \in \mathcal{B}^{j+3}$ for j even, and $G_j \in \mathcal{B}^{j+2}$ for j odd. The coefficients G_j can be computed explicitly. In particular,

$$G_{-2} = \mathcal{P} = 0, \quad G_{-1} = \mathcal{P} \neq 0.$$

Theorem 3.3. *Assume that the threshold 0 is an exceptional point of the second kind, and that Assumption 2.1 is fulfilled for some integer $\beta \geq 4$. Then*

$$R(\kappa) = \sum_{j=-2}^{\beta-6} \kappa^j G_j + \mathcal{O}(\kappa^{\beta-5}) \quad \text{in } \mathcal{B}^{\beta-2}$$

with $G_j \in \mathcal{B}^{j+3}$ for j even, and $G_j \in \mathcal{B}^{j+2}$ for j odd. The coefficients G_j can be computed explicitly. In particular,

$$G_{-2} = \mathcal{P} \neq 0, \quad G_{-1} = \mathcal{P} = 0.$$

Theorem 3.4. *Assume that the threshold 0 is an exceptional point of the third kind, and that Assumption 2.1 is fulfilled for some integer $\beta \geq 4$. Then*

$$R(\kappa) = \sum_{j=-2}^{\beta-6} \kappa^j G_j + \mathcal{O}(\kappa^{\beta-5}) \quad \text{in } \mathcal{B}^{\beta-2}$$

with $G_j \in \mathcal{B}^{j+3}$ for j even, and $G_j \in \mathcal{B}^{j+2}$ for j odd. The coefficients G_j can be computed explicitly. In particular,

$$G_{-2} = \mathcal{P} \neq 0, \quad G_{-1} = \mathcal{P} \neq 0.$$

Theorems 3.1–3.4 justify the classification of threshold types only by the growth properties of eigenfunctions:

Corollary 3.5. *The threshold type determines and is determined by the coefficients G_{-2} and G_{-1} from Theorems 3.1–3.4.*

We can also compute the coefficients G_0 and G_1 . They can be considered as part of the main results of [IJ3]. However, their expressions are very long, and we omit them in this article, see [IJ3, Appendix B]. These results are generalizations of [IJ1] on the discrete full line \mathbb{Z} and [IJ2] on the discrete half-line \mathbb{N} . The strategy for proofs is also similar to [IJ1, IJ2], implementing the expansion scheme of [JN1, JN2] in its full generality. However, due to our choice of the free operator the expansion procedure gets simplified.

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